

Thermal transport properties of layered composites by transient measurement : Identification by a new numerical algorithm

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Transient methods are widely used to determine the thermal transport properties of dielectrics and other materials. In some situations they can be used in case of homogeneous media to measure several properties either simultaneously or separately. In this context analytical model are available and a well-posed inverse problem of parameter identification has to be solved. The examination of composite media is more complicated and only a few results are known. The algorithm proposed here allows to simultaneously determine the thermal conductivity and thermal diffusivity of layered dielectrics by transient measurements. It is based on a plane source which acts both as a resistive heater and temperature sensor. For the technique to be successful two essential aspects have to be considered: firstly, the mathematical modeling of the measured data (the forward problem) and secondly, the problem of ill-posedness of the inverse problem. For the proposed measurement configuration, a new fast data analysis algorithm based on an analytic solution for the forward problem is presented. In principle, a numerical solution such as FEM solution of the heat conduction equation can be used instead of the analytical one, but the computational effort is much greater. The inverse problem is formulated as an output-least squares problem, which leads to a transcendent algebraic system of equations. The method was successfully tested for different situations.

1 INTRODUCTION

Information on thermal transport properties has increasingly gained in importance in the fields of engineering which try to reduce the energy involved, e.g., in process engineering and in the building industry. In the case of homogeneous media, besides classical steady-state methods, alternative transient techniques are now becoming widely used worldwide for all types of material (cf. [1] – [5]). The thermal conductivity λ and the thermal diffusivity a are derived quantities and, thus, cannot be measured directly. They rather have to be determined from related quantities, e.g., a temperature profile. In general, a heat flow of known rate, Φ , is passed through the material under test and the associated temperature profile $T(\mathbf{x}, t)$ is measured depending on the thermal properties. In several situations, these methods can be used to measure several properties either simultaneously or separately. In this context a well-posed inverse problem of parameter identification has to be solved.

The examination of composite media has only recently been considered with only a few results; cf. [6]-[9]. Analytic approximations of the solution of the forward problem, that means the simulation of the measuring signal are available for homogeneous media, which in general loses its validity for layered composites. A way out is the application of numerical methods as finite-element and finite-difference methods but at the expense of considerable computational effort.

For the measurement configuration proposed here, a new fast data analysis is derived on the basis of an analytic solution of the heat conduction equation with piecewise-constant thermal transport properties corresponding to the separate layers. It allows the thermal conductivity

and thermal diffusivity of layered dielectrics to be simultaneously determine by transient measurements. A plane heat source consisting of thin metal foil acts both as a resistive heater and as a temperature sensor.

In the first part of Section 2, the mathematical model for the transient temperature distribution applied to the proposed method is given. An analytic expression describing the measuring signal based on the Green's function formulation is derived in the second part. The section ends with the output-least squares algorithm for the inverse problem, which leads to a transcendent algebraic system of equations.

In Section 3, the theoretical ideas of Section 2 are successfully applied to a layered sample where the algorithm is split into two steps. In the first step, the properties of the inner core and in the second step, those of the outer layer are identified. Summary is given in Section 4.

2 THEORY

2.1. The mathematical model

The data analysis of the implicit measuring method consists of two parts: The first part is the so-called forward problem for which a mathematical model is derived relating the measurement data to the thermal properties; this means that for known thermal properties of the sample and a known experimental set-up, the corresponding measuring signal can be simulated. The second part is the inverse problem; for given measuring signal and a known experimental set-up, the thermal transport properties have to be identified.

The principle of the method proposed is shown in Figure 1. A current-carrying metallic foil is clamped between two layered sample halves and simultaneously acts as a temperature sensor. This method works similar to the hot strip technique: Both are based on a step-wise heat source which is combined with the temperature sensor. The difference consists in the heat source geometry, on the one hand a plane source and on the other a hot strip. In another conceivable version, the temperature response could be measured by a separate sensor placed a distance h from the heat source but within the inner layer. It describes a step-wise transient technique based on separated heat source and temperature sensor.

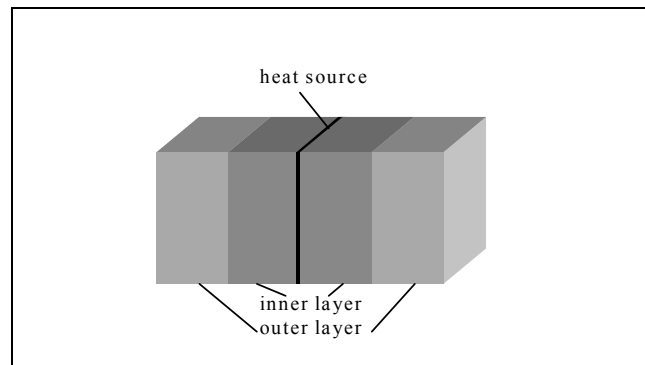


Figure 1. Schematic diagram of the specimen for the proposed method

The choice of a plane source geometry provides the possibility of modeling the heat transfer process as a one-dimensional problem. Even in case of multi-layered composites, an analytic solution of the corresponding partial differential equation for heat transfer can be derived as a solution of the forward problem. The length and width of the foil have to be sufficiently large for the heat losses at the surface to be negligible. Another way allowing one-dimensional

modeling of the temperature response may realized by total insulation at the surface, i.e. the assumption of homogeneous boundary condition of the second kind.

The one-dimensional formulation of the transient heat conduction problem for a m -layered slab is given as follows. The interfaces between the layers are located at $x = x_i$, $i = 1, 2, \dots, m-1$ and the outer boundary surfaces at x_0 and x_m . Let λ_i be the thermal conductivity and a_i the thermal diffusivity of the i -th layer, $x_{i-1} < x \leq x_i$, $i = 1, \dots, m$. We get the differential equation

$$\frac{\partial T(x, t)}{\partial t} = \text{div} (a \mathbf{grad} T(x, t)) + \frac{a}{\lambda} q(x, t) \text{ in } x_0 \leq x \leq x_m, t > 0. \quad (1)$$

The heat source q is restricted to the thin foil and the thermal diffusivity a and the thermal conductivity λ depend on the diffusivities and conductivities of the single layers

$$a(x) = a_i, \quad \lambda(x) = \lambda_i, \quad x_{i-1} < x \leq x_i.$$

The initial temperature at $t = 0$

$$T(x, 0) = T_0(x), \quad x_0 \leq x \leq x_m \quad (2)$$

can be different for each layer and also vary within a layer. At the outer boundaries we write the general form

$$-\lambda_1 \frac{\partial T(x_0, t)}{\partial x} = h_0 (T_1(t) - T(x_0, t)) \quad , t > 0 \quad (3a)$$

$$\lambda_m \frac{\partial T(x_m, t)}{\partial x} = h_m (T_m(t) - T(x_m, t)) \quad , t > 0, \quad (3b)$$

where $T_1(t)$ and $T_m(t)$ stand for the– possibly time-dependent – ambient temperature of the first and the m -th (last) layer, respectively. Additional boundary conditions at the layer interfaces have to be satisfied,

$$\lambda_i T_x(x_i-0, t) = \lambda_{i+1} T_x(x_i+0, t) \quad (3c)$$

ensuring the continuity of heat flux at the interfaces. In the case of perfect thermal contact between the layers, we have the continuity of temperature

$$T(x_i - 0, t) = T(x_i + 0, t), \quad i = 1, 2, \dots, m-1, \quad (3d)$$

and in case there is thermal contact conductance at the interfaces,

$$-\lambda_i T_x(x_i, t) = h_i (T(x_i - 0, t) - T(x_i + 0, t)), \quad i = 1, 2, \dots, m-1, \quad (3d)^*$$

with the interface thermal contact conductance h_i at the i -th interface.

Let us adapt the general mathematical model to our special experimental situation. Due to the symmetrical set-up of a centered inner source, the integration domain of equation (1) can be reduced to one half. Furthermore, concentrating on a sample simply consisting of two materials with perfect thermal contact we get the following conditions for our specialized problem with $m=2$ and $x_0=0$ corresponding to (3a–d)

$$-\lambda_1 \frac{\partial T(0,t)}{\partial x} = 0 \quad , t > 0 \quad (4a)$$

because of symmetry the heat flux vanishes

$$\lambda_2 \frac{\partial T(x_2,t)}{\partial x} = h_2 (T_2(t) - T(x_2,t)) \quad , t > 0 \quad (4b)$$

the outer boundary condition and the conditions at the interface

$$\lambda_1 T_x(x_1-0,t) = \lambda_2 T_x(x_1+0,t) \quad (4c)$$

$$T(x_1-0,t) = T(x_1+0,t) \quad (4d)$$

If d is half the thickness of the foil, we get the following form for the heat source q

$$q(x,t) = q(x) = \begin{cases} q_0, & x \leq d \ll x_1 \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

In reality, the materials and thus the thermal properties of the thin foil (10-20 μm) and the inner layer are highly different. Considering the foil as a separate layer or neglecting the separate layer leads to a small difference in the calculated temperature distribution. Nevertheless, this difference is covered by the measurement uncertainty as shown in [10] by finite-element simulations. For simplicity, the different material properties of the foil are neglected in the following. Then, the mathematical formulation is given by the heat conduction equation

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} \left(a \frac{\partial T(x,t)}{\partial x} \right) + \frac{a}{\lambda} q(x) \quad \text{in } 0 \leq x \leq x_2, t > 0 \quad (6)$$

with the initial condition (2), the boundary conditions (4a–b), the interface conditions (4c–d) and the heat source $q(x)$ given in (5).

2.2. Analytic solution of the forward problem

To solve the nonhomogeneous heat transfer problem (6) in a layered composite medium, we start with homogeneous one with no heat generation

$$\frac{\partial \bar{T}(x,t)}{\partial t} = \frac{\partial}{\partial x} \left(a \frac{\partial \bar{T}(x,t)}{\partial x} \right) \quad \text{and } \bar{T}(x,0) = T_0(x) \quad \text{in } 0 \leq x \leq x_2, t > 0 \quad (7)$$

to obtain the Green's function. When this function is available, the temperature distribution of (6) can be represented only in terms of Green's function.

Assume a separation of variables in space- and time-dependent functions in the form

$$\bar{T}(x,t) = u(x)\Gamma(t) \quad (8)$$

The time-variable function $\Gamma(t)$ is the solution of

$$\frac{d\Gamma(t)}{dt} + \beta_n^2 \Gamma(t) = 0 \quad \text{for} \quad t > 0$$

given by

$$\Gamma_n(t) = e^{-\beta_n^2 t}. \quad (9)$$

The corresponding eigenvalue problem is given by

$$\frac{d^2 u_i}{dx^2} + \frac{\beta_n^2}{a_i} u_i = 0 \quad (10)$$

with the dimensionsless split eigenfunctions

$$u_n(x) = \begin{cases} u_{1n}(x), & \text{for } 0 \leq x \leq x_1 \\ u_{2n}(x), & \text{for } x_1 < x \leq x_2 \end{cases}$$

where the index n indicates the dependence on the eigenvalue β_n . The solution is subject to the boundary conditions

$$-\lambda_1 \frac{du_{1n}(x)}{dx} = 0 \quad \text{at } x = 0 \quad (11a)$$

$$u_{1n}(x) = u_{2n}(x) \quad \text{at } x = x_1 \quad (11b)$$

$$\lambda_1 \frac{du_{1n}}{dx} = \lambda_2 \frac{du_{2n}}{dx} \quad \text{at } x = x_1 \quad (11c)$$

$$\lambda_2 \frac{du_{2n}}{dx} = 0 \quad \text{at } x = x_2 \quad (11d)$$

where we assume adiabatic conditions at the sample surface. The general solution u_{in} of the eigenvalue problem (10) for a slab geometry can be written in the form

$$u_{1n}(x) = A_{1n} \sin\left(\frac{\beta_n}{\sqrt{a_1}} x\right) + B_{1n} \cos\left(\frac{\beta_n}{\sqrt{a_1}} x\right) \quad \text{in } 0 \leq x \leq x_1 \quad (12a)$$

$$u_{2n}(x) = A_{2n} \sin\left(\frac{\beta_n}{\sqrt{a_2}} x\right) + B_{2n} \cos\left(\frac{\beta_n}{\sqrt{a_2}} x\right) \quad \text{in } x_1 < x \leq x_2 \quad (12b)$$

The first boundary condition (11a) requires that $A_{1n}=0$. Without loss of generality one of the nonvanishing coefficients can be set to unity since one coefficient is arbitrary. We have chosen $B_{1n}=1$. Moreover, the solution (12) has to fulfill the remaining conditions (11b-d) yielding the following equation in matrix form for the determination of the coefficients A_{2n} and B_{2n}

$$\begin{bmatrix} \cos\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right) & -\sin\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) & -\cos\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) \\ -\frac{\lambda_1}{\lambda_2}\sqrt{\frac{a_2}{a_1}}\sin\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right) & -\cos\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) & \sin\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) \\ 0 & \cos\left(\frac{x_2\beta_n}{\sqrt{a_2}}\right) & -\sin\left(\frac{x_2\beta_n}{\sqrt{a_2}}\right) \end{bmatrix} \begin{bmatrix} 1 \\ A_{2n} \\ B_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

The matrix of coefficients in equation (13) depends on the eigenvalues β_n of the problem (10), unknown so far. Nevertheless, they are determined by the requirement of the vanishing determinant, the condition for the existence of a solution of equation (13). After having determined the β_n , in general by numerical methods, we obtain for A_{2n} and B_{2n}

$$A_{2n} = \cos\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right)\sin\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) - \frac{\lambda_1}{\lambda_2}\sqrt{\frac{a_2}{a_1}}\sin\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right)\cos\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) \quad (14a)$$

$$B_{2n} = \frac{\lambda_1}{\lambda_2}\sqrt{\frac{a_2}{a_1}}\sin\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right)\sin\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right) + \cos\left(\frac{x_1\beta_n}{\sqrt{a_1}}\right)\cos\left(\frac{x_1\beta_n}{\sqrt{a_2}}\right). \quad (14b)$$

Now, with (14) and the knowledge of the eigenvalues β_n the eigenfunctions u_{in} defined in equation (12) are known and the general solution of (7) becomes

$$\bar{T}_i(x, t) = \sum_{j=1}^2 \int_{x_{j-1}}^{x_j} \left[\sum_{n=1}^{\infty} \frac{1}{N_n} \frac{\lambda_j}{a_j} e^{-\beta_n^2 t} u_{in}(x) u_{jn}(x') \right] T_0(x') dx' \quad , i = 1, 2 \quad (15)$$

where

$$\bar{T}(x, t) = \begin{cases} \bar{T}_1(x, t), & \text{for } 0 \leq x \leq x_1 \\ \bar{T}_2(x, t), & \text{for } x_1 < x \leq x_2 \end{cases}$$

and the norm N_n is given by

$$N_n = \frac{\lambda_1}{a_1} \int_0^{x_1} u_{1n}^2(x) dx + \frac{\lambda_2}{a_2} \int_{x_1}^{x_2} u_{2n}^2(x) dx \quad . \quad (16)$$

The expression in brackets in equation (15) is the Green's function for the homogeneous problem. By replacing t by $(t-\tau)$ we obtain the Green's function G_{ij} for the composite medium for the nonhomogeneous case

$$G_{ij}(x, t | x', \tau) = \sum_{n=1}^{\infty} e^{-\beta_n^2(t-\tau)} \frac{1}{N_n} \frac{\lambda_j}{a_j} u_{in}(x) u_{jn}(x') \quad (17)$$

$$x_{i-1} < x \leq x_i, i=1, 2, \quad x_{j-1} < x' \leq x_j, j=1, 2 \quad .$$

It represents the response at location x and at time t to an impulse located at x' at time τ . There are an infinite number of discrete eigenvalues β_n and the corresponding eigenfunctions u_{in} .

The eigenvalues are ordered by magnitude $\beta_1 < \beta_2 < \dots < \beta_n < \dots$. For more details, see references [11] – [13]. Finally, the complete temperature distribution in the sample can be calculated by the resulting formula, where $T_1(x, t)$ stands for the temperature in the inner layer and $T_2(x, t)$ for that in the outer layer

$$T_i(x, t) = \sum_{j=1}^2 \left\{ \int_{x_{j-1}}^{x_j} G_{ij}(x, t | x', \tau) \Big|_{\tau=0} T_0(x') dx' + \int_0^t \int_{x_{j-1}}^{x_j} G_{ij}(x, t | x', \tau) \frac{a_j}{\lambda_j} q(x', \tau) dx' d\tau \right\} \quad (18)$$

in $x_{i-1} < x \leq x_i, i=1, 2$

We assume, that the initial temperature in the sample at $t = 0$ is constant, $T_0(x) = T_0$, as can be expected for the experimental configuration. Then, using the substitution

$$\Delta T_i(x, t) = T_i(x, t) - T_0 \quad (19)$$

for the temperature rise ΔT_i , the derivation of the solution is very similar, but in the resulting expression, the first integral of (18) vanishes. Remember that the constant heat source (5) is limited to a thin heater, in the one-dimensional model to a short interval, and the solution becomes

$$\Delta T_i(x, t) = q_0 \frac{a_1}{\lambda_1} \int_0^t \int_0^d G_{i1}(x, t | x', \tau) dx' d\tau \quad \text{in } x_{i-1} < x \leq x_i, i=1, 2. \quad (20)$$

The transient signal, measured, i.e. the temperature rise in the source plane, is calculated from (20) and the derived Green's function (17) at $x = 0$ to be

$$\Delta T_1(0, t) = q_0 \sum_{n=1}^{\infty} \frac{\sqrt{a_1}}{\beta_n^3 N_n} \left[1 - e^{-\beta_n^2 t} \right] \sin \left(\frac{\beta_n}{\sqrt{a_1}} d \right) \quad (21)$$

Adding the initial temperature T_0 at $t = 0$ corresponding (19) we get the temperature signal $T_1(0, t)$, the solution of the forward problem.

2.3. The inverse problem

The objective is to find the thermal conductivity and diffusivity values of the two-layered composite under test which are consistent with the experimental measuring signal. The underlying mathematical model relating the experimental set-up and the thermal properties of the sample with the measuring signal is generally given by equations (4) – (6). The explicit form, equation (21), has been derived in the last section. This analytic solution has the advantage of requiring very little computing time compared with numerical solution methods as finite-element or finite-difference methods. On the other hand, the experimental configurations for which analytic expressions can be derived simulating the measuring signal in multi-layered samples are limited to special cases.

Let the vector $\mathbf{T}^{sim}(\lambda_1, a_1, \lambda_2, a_2, \mathbf{t})$ be the simulated measuring signal depending on the thermal properties and discrete times $\mathbf{t} = (t_1, \dots, t_s)$, and $\mathbf{T}^{mes}(\mathbf{t})$ the vector of the measuring signal. The related inverse problem of parameter identification is formulated as an output-least squares problem

$$\|T^{sim}(\lambda_1, a_1, \lambda_2, a_2, t) - T^{mes}(t)\|_2^2 = \min!$$

based on the repeated solving of the forward problem, in conjunction with a minimization strategy. The subscript indicates the l_2 -norm. It is solved by the Levenberg-Marquardt method [14] as published by the program library of the International Mathematical Subroutine Library (IMSL). The algorithm combines the Gauss-Newton method with the gradient method which is well suited for handling ill-conditioned problems. This procedure was also successfully applied to homogeneous and multi-layered problems, solving the forward problem by a finite-element method [8] , [9].

3. NUMERICAL EXPERIMENTS

Now, we test the proposed technique by reconstructing thermal transport properties of a layered sample obtained from simulated data. The geometrical dimensions and the thermal properties of the sample are chosen as follows:

Thickness of the heater $2d = 20\mu\text{m}$, thickness of the inner layer $d_1 = 20\text{ mm}$, thickness of the outer layer $d_2 = 20\text{ mm}$, thermal properties of the inner layer $\lambda_1 = 1.5\text{ Wm}^{-1}\text{K}^{-1}$ and $a_1 = 1.0\text{ mm}^2\text{s}^{-1}$ and of the outer layer $\lambda_2 = 0.5\text{ Wm}^{-1}\text{K}^{-1}$ and $a_2 = 0.5\text{ mm}^2\text{s}^{-1}$.

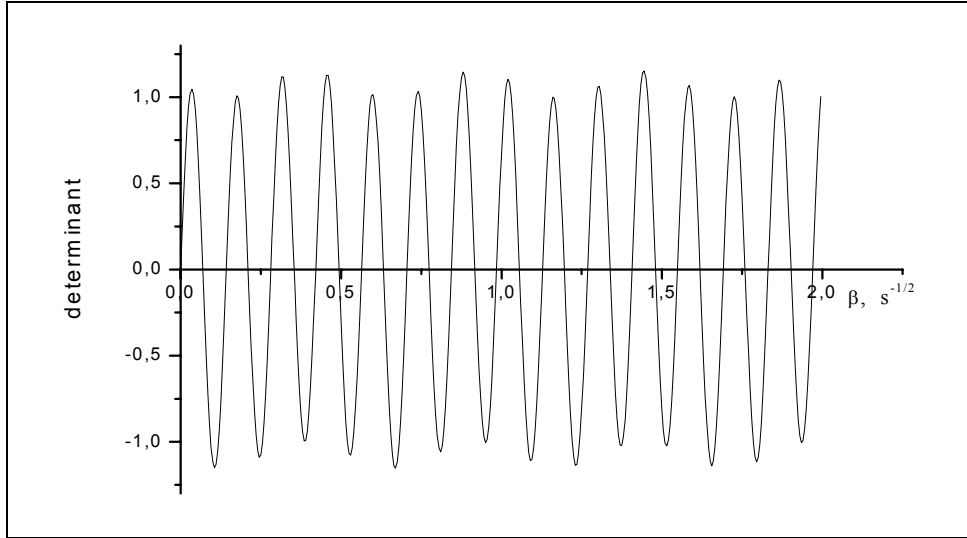


Figure 2. The determinant of the coefficients given in equation (13) versus β .

First, the matrix of coefficients is calculated given in equation (13). From the requirement that the determinant of the coefficients must vanish, we determine the eigenvalues of the corresponding eigenvalue problem (10). Figure 2 shows the determinant as a function of β where the zeros are the wanted eigenvalues. A precise calculation was achieved using a subroutine of the IMSL library for the determination of zeros of transcendental functions and furnishes an arbitrary number of eigenvalues β_n . The current number r needed depends on the convergence properties of the series in equation (21). In our example, the condition $\beta_i < 1.5$ is sufficient, and up to $r = 21$ eigenvalues have to be taken into account; their values are listed in Table 1.

0.0692	0.5640	1.0562
0.1424	0.6322	1.1279
0.2103	0.7055	1.1955
0.2813	0.7743	1.2683
0.3535	0.8440	1.3384
0.4210	0.9170	1.4069
0.4935	0.9847	1.4802

Table 1. The eigenvalues of equation (10) smaller than 1.5.

The corresponding measuring signal calculated by (21) as well the signal for a homogenous sample (λ_1 and a_1) are shown in Figure 3. In the first interval $[0, t_z]$, only the thermal properties of the inner core govern the temperature rise. Therefore, the curves of the two-layered and the homogeneous sample coincide. For $t > t_z$ the temperature rise is determined by the thermal conductivity and diffusivity of both the inner core and the outer layer as well at the later times also by the surroundings.

As expected from earlier investigations using the finite-element method for the forward problem of the hot strip technique, the inverse problem of the simultaneous identification of the four properties $\lambda_1, a_1, \lambda_2$ and a_2 is highly ill-posed. To improve the condition of the problem, the thermal properties have to be determined one after another, starting with the inner layer. As a first step, the initial interval $[0, t_z]$ of the signal is selected to identify the

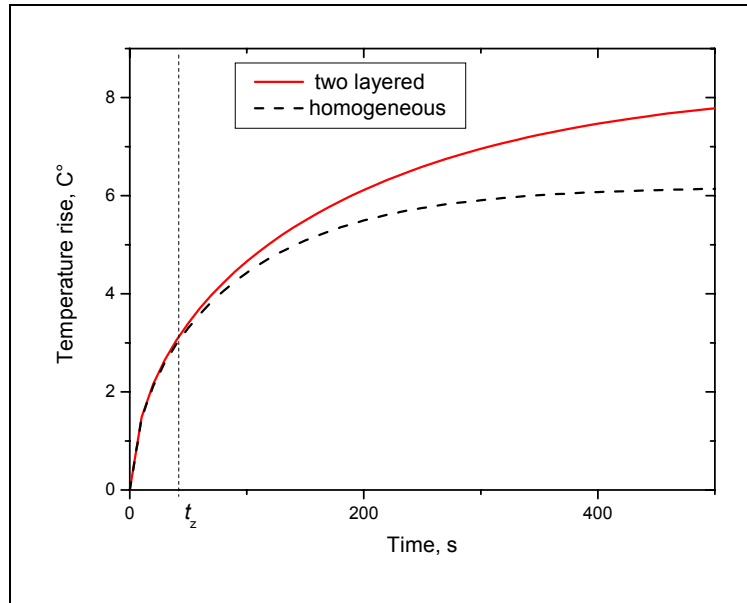


Figure 3. Calculated temperature rise for a layered sample and a corresponding homogeneous sample (a_1, λ_1).

thermal properties of the inner layer. In a subsequent step, the determination of the properties of the outer layer is carried out within the remaining interval ($t > t_z$) keeping the results of the first step fixed. This procedure substantially improves the accuracy of the result to the exact values.

In principle, the method can be extended to more than two layers. Nevertheless, a worsening of the condition of the inverse problem can be expected, resulting in a higher uncertainty of the results.

4. SUMMARY

Transient methods are widely used to determine the thermal properties of some materials, but almost always for homogeneous media. For the situation of layered composites a new fast identification algorithm is presented which is based on an analytic solution of the forward problem and a numerical least-squares solver. For the measuring configuration, a plane source is favored because a one-dimensional treatment is possible in this case. The method is designed for simultaneous determination of the four properties, viz. thermal diffusivity and conductivity of the two layers.

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